BINOMIAL THEOREM

Pascal Triangle

$(a+b)^0 \rightarrow 1$

$(a+b)^1 \rightarrow 1 \quad 1$

$(a+b)^2 \rightarrow 1 \quad 2 \quad 1$

$(a+b)^3 \rightarrow 1 \quad 3 \quad 3 \quad 1$

$(a+b)^4 \rightarrow 1 \quad 4 \quad 6 \quad 4 \quad 1$

$(a+b)^5 \rightarrow 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

Coefficients of Variables

Binomial Theorem

- Let n be any natural number and x, a be any real number, then

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots$$

$$+ {}^nC_r\, x^{n-r} a^r + \ldots\ldots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

$$\Rightarrow (x+a)^n = \sum_{0}^{n} {}^nC_r x^{n-r} a^r \quad \text{where } {}^nC_r = \frac{n!}{r!(n-r)!}$$

Some Important Properties of C

${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$	${}^nC_r = n/(n-r).{}^{n-1}C_r$
${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$	${}^nC_r = {}^nC_{n-r}$

- Total number of terms of expansion = n+1
- sum of the indices of x and a in each term is n.

Some Important Results of Binomial Theorem

• On replacing a by −a	$(x-a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^{n-r} a^r$
• Putting x = 1 and a = x	$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$
• Putting a = 1	$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}$
• Putting x = 1 and a = −x	$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^r\, {}^nC_r x^r + \ldots + (-1)^n\, {}^nC_n x^n$

- Further Results into

$$(x+a)^n + (x-a)^n = 2\left[{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots\right]$$

  - no. of terms
    - if n is even : (n/2) + 1
    - if n is odd : (n+1)/2

$$(x+a)^n - (x-a)^n = 2\left[{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots\right]$$

  - no. of terms
    - n is even or odd : (n+1)/2

- General term in any expansion is T_{r+1}

Properties of the expansion $(1+x)^n$

1. Coefficients of terms equidistant from the beginning and the end are equal.
2. Sum of the binomial coefficients is 2^n.
3. Sum of coefficient of odd terms = sum of coefficient of even terms = 2^{n-1}
4. $C_0 - C_1 + C_2 - C_3 + C_4 - \ldots\ldots + (-1)^n C_n = 0$

Middle term in a binomial Expression

If n is even (n/2)+1th term	$T_{\left(\frac{n}{2}+1\right)} = {}^nC_{\frac{n}{2}} x^{\frac{n}{2}} a^{\frac{n}{2}}$
If n is odd (n+1)/2th (n+3)/2th	$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\left(\frac{n-1}{2}\right)} x^{\frac{n+1}{2}} a^{\frac{n-1}{2}}$
	$T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\left(\frac{n+1}{2}\right)} x^{\frac{n-1}{2}} a^{\frac{n+1}{2}}$

Binomial coefficient of middle term is the greatest.

Which is Larger?

- e.g. $(1.02)^{4000}$ or 800
  - $(1+0.2)^{4000} = 1 + {}^{4000}C_1 \times 0.2 + {}^{4000}C_2 (0.2)^2 + \ldots$
    $= 1 + 800 + \text{Positive Values} > \mathbf{800}$

Finding Greatest Term

- To find greatest term in the expansion of $(a + bx)^n$, use the following algorithm
  a. Put $(T_{r+1}/T_r) = 1$
  b. if r is a non-integer, greatest term is $[T_{r+1}]$ where [ ] = greatest integer function.
  c. if r is integer, $T_{r+1} = T_r$, both greatest

Divisibility Problems Trick

Remember that $(1+x)^n - 1 - nx$ is divisible by x^2

- e.g. $6^{2n} - 35n - 1$ is divisible by 35^2
  - $(6^2)^n - 35n - 1 = 36^n - 35n - 1 = (1+35)^n - 1 - 35n$
  - The above is divisible by 35^2

- e.g. Find remainder when 3^{63} is divided by 26
  - $3^{63} = (27)^{21} = (1+26)^{21} = 1 + {}^{21}C_1.26 + {}^{21}C_2.26^2 + \ldots$
  - $1 + 26(k) \Rightarrow$ Remainder = 1

- e.g. Find last two digits of $(27)^{27}$ (take nearest multiple of 10)
  - $(27)^{27} = 3(81)^{20} = 3(80+1)^{20}$
  - $= 3[1 + {}^{20}C_1.80 + {}^{20}C_2.(80)^2 + {}^{20}C_3.(80)^3 + \ldots.. + {}^{20}C_{20}.(80)^{20}]$
  - $= 3[1 + 1600]$ + multiple of 100
  - $= 4803$ + multiple of 100
  - i.e. 03 last two digits

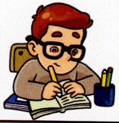
Binomial theorem for any index

- n be a rational number and x be a real number

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots + \text{terms upto } \infty$$

- General term for expansion

$$\frac{n(n-1)(n-2)\dots\dots\dots(n-r+1)}{r!}x^r$$



Multinomial Theorem

$$(x_1 + x_2 + \dots\dots + x_k)^n = \sum_{r_1+r_2+\dots\dots r_k = n} \frac{n!}{r_1!r_2!\dots\dots r_k!} x_1^{r_1} x_2^{r_2} \dots\dots x_k^{r_k}$$

Total Number of Terms	$(x_1 + x_2 + \dots\dots x_n)^m$ is ${}^{m+n-1}C_{n-1}$